The linear approximation equations (6.1) are in this case

$$y_1[k+1] = v[k], \quad y_2[k+1] = y_1[k] \tag{7.3}$$

The first equation of system (6.2) may be written

$$\beta(x)(x_1x_2 + u) = v, \quad \beta(x) = (x_2^2+1)^2x_1^2 + 1 \tag{7.4}$$

and the second becomes an identity. Under the control $v = D(y) = ay_1 + by_2, \quad a = 1,\ b = -0.25$ system (7.3) is asymptotically stable in the large. By (7.4), the isomorphism yields a control

$$u = (ay_1 + by_2)\beta^{-1}(x) - x_1x_2 = [a(x_2^2+1)x_1 + bx_2]\beta^{-1}(x) - x_1x_2$$

of system (7.1), under which, by Corollary 6.1, the trivial solution is asymptotically stable in the large. The same control is also obtainable using the following formula form /4/:

$$u = \lim_{\lambda \to 0} \lambda^{-1} \odot_u D(\lambda \odot_x x)$$

REFERENCES

1. PANASYUK A.I., The introduction of algebraic operations on the set of trajectories of a non-linear control system. Prikl. Mat. Mekh., 42, 4, 1984.
2. KOSTRIKIN A.I. and MANIN YU.L., Linear Algebra and Geometry, Nauka, Moscow, 1986.
3. PANASYUK A.I., Algebraic operations compatible with the dynamics of continuous non-linear control systems. Differents. Uravn., 24, 3, 1988.
4. PANASYUK A.I., A method of synthesizing discrete control systems by introducing algebraic operations on the set of trajectories. Izv. Vuz. Priborostroyenie, 9, 1986.

Translated by D.L.

PMM U.S.S.R., Vol.52, No.5, pp.579-585, 1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

DYNAMIC MODELLING OF UNKNOWN PERTURBATIONS IN PARABOLIC VARIATIONAL INEQUALITIES*

V.I. MAKSIMOV

The problems involved in the dynamic determination of the load acting on a membrane rigidly fixed on a horizontal frame are investigated, and the thermal flux in a thermostat is determined. These problems are treated as special cases of a more general problem: dynamic modelling of unknown characteristics in parabolic variational inequalities. The problem is solved by constructing an algorithm, stable to information noise and computing errors, based on methods of positional control theory /1, 2/. This algorithm may be regarded as a modification of an algorithm proposed in /3/ for control systems described by ordinary differential equations. A model problem is solved. The research reported, here relies on /3, 4/ and is a sequel to /5/.

1. In /6/ (Vol.1, p.198) a numerical method of determining the deflection $y(x,t)$ of a membrane rigidly fixed on a horizontal frame with constant tension F, subject to a given load $g(x,t)$ is proposed.

We shall consider the inverse problem: to determine the load $g(x,t)$ given the deflection $y(x,t)$. Let Ω be the plane region bounded by the frame. Put

$$u(x,t) = g(x,t)/F, \quad K = \{v(\cdot) \mid v(x) \in H_0^1, \quad v(x) \leqslant 0 \text{ in } \Omega\}$$
$$K_* = \{v(\cdot) \mid v(\cdot) \in L_2([t_0,\vartheta]; H_0^1),\ \partial v/\partial t \in L_2([t_0,\vartheta]; H^{-1}),$$
$$v(t_0) = y_0\}$$

$H_0^1(\Omega)$ and $H^{-1}(\Omega)$ are Sobolev spaces. The deflection process for a membrane subject to a

*Prikl.Matem.Mekhan., 52, 5, 743-750, 1988

load $g(x, t)$ is described by the following inequality, where Δ is the Laplace operator:

$$\int_\Omega (\partial y(x,t)/\partial t - \Delta y(x,t) - u(x,t), v(x) - y(x,t))\, dx \geqslant 0, \ \ y(t) \in K$$

for a.e. $t \in [t_0, \vartheta]$ and all $v(\cdot) \in K$, $y(\cdot) \in K$.

Let us assume that the tension F is given, but the load $g(x, t)$ acting during the time interval $[t_0, \vartheta]$ is unknown. At time $\tau_i \in [t_0, \vartheta]$, $\tau_i = t_0 + i\delta$, $\delta > 0$ the deflection $y(x, \tau_i)$ is measured to some accuracy, i.e., one has a function $\psi(x, \tau_i)$ approximating $y(x, \tau_i)$. The problem is to devise a dynamic algorithm, operating in real time, to calculate $g(x, t)$. This is a meaningful formulation of the problem of the dynamic determination of the load. Measurement of the deflection at every point $x \in \Omega$ may involve technical difficulties. In that case it is natural to propose the problem of calculating g, given the deflections $y(x, \tau_i)$ measured at discrete points $x = x_j \in \Omega$.

Let us agree on the following notation: H, V and U are real Hilbert spaces with norms $|\cdot|, \|\cdot\|$ and $\|\cdot\|_V$, $V \subset H$, V is continuously and densely embedded in H, $(\cdot,\cdot)$ is the scalar product in H, $(\cdot,\cdot)_{V\times V^*}$ is the duality between V and V^*; $L(U, X)$ is the Banach space of continuous linear operators from U to X; $W^{1,2}([t_0, \vartheta]; H)$, $C([t_0, \vartheta]; H)$ and $L_2([t_0, \vartheta]; H)$ are the standard spaces; $W^{1,2}((t_0, \vartheta]; H)$, $(L_2((t_0, \vartheta]; H))$ is the space of functions $y(\cdot)$: $(t_0, \vartheta] \to H$ such that $y(\cdot)$: $[t_0 + \varepsilon, \vartheta] \to H$ is an element of $W^{1,2}([t_0 + \varepsilon, \vartheta]; H)$ $(L_2([t_0 + \varepsilon, \vartheta]; H))$ for every $\varepsilon \in (0, \vartheta - t_0)$; Δ is a partition of the interval $[t_0, \vartheta]$, of mesh $\delta = \delta(\Delta)$ (i.e., the set of points $\{\tau_i\}$, $i \in [0: m]$, $m = m(\Delta)$, $t_0 = \tau_0 < \tau_1 < \ldots < \tau_m = \vartheta$, $\tau_i = \tau_{i-1} + \delta$); U_0 is the set of admissible controls; $\bar{R} = R \cup \{+\infty\}$; $\bar{A}$ is the closure of $A \subset H$, $A_1 - A_2 = \{x - y \mid x \in A_1, y \in A_2\}$; $D(B)$ and $R(B)$ are the domain and range of an operator B; $\partial\varphi$ is the subdifferential of φ; $\partial\varphi_\lambda$ is the Iosida operator /7/ of the mapping $x \to \partial\varphi(x)$ corresponding to a parameter $\lambda > 0$; $\partial\varphi^0(y) = \{z \in H \mid |z| = \inf |x|, \ x \in \partial\varphi(y)\}$; $u_{a,b}(\cdot)$ is the function $u(t)$, $t \in [a, b]$. Throughout, integrals are understood in Bochner's sense and derivatives in the generalized sense. In particular, it follows from the properties of Iosida operators (see, e.g., /7/) that $\partial\varphi_\lambda(x) \to \partial\varphi^0(x)$ as $\lambda \to 0+$.

2. Consider the dynamic system Σ described by the parabolic variational inequality:

$$\begin{aligned}&(y^{\cdot}(t) - B(t, y(t))u(t) - f(t), y(t) - z) + (Ay(t), y(t) - \\ &z)_{V\times V^*} + \varphi(y(t)) - \varphi(z) \leqslant 0 \text{ for a.e. } t \in (t_0, \vartheta) \text{ and all} \\ &z \in V\end{aligned} \tag{2.1}$$

Here $f(\cdot) \in L_2([t_0, \vartheta]; H)$ is a given function, $A: V \to V^*$ a continuous, symmetric linear operator satisfying the coercivity condition for some $\omega > 0$ and α:

$$(Az, z)_{V\times V^*} + \alpha |z|^2 \geqslant \omega \|z\|^2 \text{ for all } z \in V$$

φ: $V \to \bar{R}$ is a convex, lower semicontinuous eigenfunction.

The family of operators $B(t, y) \in L(U, H)$, depending on the parameters $t \in [t_0, \vartheta]$ and $y \in H$, is such that for any step function $y(\cdot)$ with finitely many jumps there exists a semicontinuous operator B_y: $L_2([t_0, \vartheta]; U) \to L_2([t_0, \vartheta]; H)$ admitting the representation $\{B_y u(\cdot)\}(t) = B(t, y(t))u(t)$.

A function $y(\cdot)$: $[t_0, \vartheta] \to H$ is called a strong solution of (2.1) for control $u(\cdot) \in L_2([t_0, \vartheta]; U)$ and initial state $y_0 \in E_1 \equiv D(\varphi) \cap V$, if for a.e. $t \in [t_0, \vartheta]$ it satisfies (2.1), $y(\cdot) \in W^{1,2}([t_0, \vartheta]; H) \cap C([t_0, \vartheta]; V)$, $y(t_0) = y_0$, $A\, y(\cdot) \in L_2([t_0, \vartheta]; H)$ and

$$y^{\cdot}(t) = (-Ay(t) - \partial\varphi(y(t)) + B(t, y(t))u(t) + f(t))^\circ \text{ for a.e. } t \in [t_0, \vartheta] \tag{2.2}$$

A function $y(\cdot)$: $[t_0, \vartheta] \to H$, $y(t_0) = y_0$, is a strong solution of (2.1) for $u(\cdot) \in L_2([t_0, \vartheta]; U)$ and $y_0 \in E_2 \equiv \overline{D(\varphi) \cap V} \setminus D(\varphi) \cap V$, if $y(\cdot) \in W^{1,2}((t_0, \vartheta]; H) \cap C([t_0, \vartheta]; H)$ and it satisfies (2.1), (2.2) for a.e. $t \in [t_0, \vartheta]$

We shall consider two cases: 1) $y_0 \in E_1$, 2) $y_0 \in E_2$. It is assumed that for an initial state y_0 and control $u_p(\cdot) \in L_2([t_0, \vartheta]; U)$ acting on Σ there exists a unique strong solution $y_p(\cdot) = y(\cdot; y_0, u_p)$. This is the case, for example, when $B(t, y) = B$, φ: $H \to \bar{R}$ is a lower semicontinuous function and there exists a constant $C > 0$, independent of $\lambda > 0$, such that for all elements $y \in D(A_H) = \{y \in V \mid Ay \in H\}$ and all operators $\partial\varphi_\lambda(y)$

$$(Ay, \partial\varphi_\lambda(y)) \geqslant -C(1 + |\partial\varphi_\lambda(y)|)(1 + |y|), \ \ \lambda > 0$$

The problem discussed in this paper is as follows. We have a system Σ subject to an unknown control $u_p(\cdot) \in U_0 \subset L_2([t_0, \vartheta]; U)$, which generates the actual motion $y_p(\cdot)$ - a strong solution of (2.1). The motion $y_p(\cdot)$ itself and the point y_0 are also unknown. However, information is available concerning the realization $y_p(\cdot)$ - signals $\psi_{\tau_{i-1}, \tau_i}(\cdot)$ received at times τ_i, $i \geqslant 1$, which are step functions with a finite number of jumps.

These signals satisfy one of the following conditions:

1) for all $i \geqslant 1$

$$|y_p(t) - \psi(t)| \leqslant \varepsilon, \quad t \in (\tau_{i-1}, \tau_i] \tag{2.3}$$

$$\int_{\tau_{i-1}}^{\tau_i} |Ay_p(t) - A\psi(t)|\, dt \leqslant \varepsilon \tag{2.4}$$

2) For all $i \geqslant 1$, only inequalities (2.3) hold, but the quantities $|Ay_p(t) - A\psi(t)|$ may take arbitrary values. Moreover, at a time $t = t_0$ we know U_0 and the element $\psi(t_0) \in H$, $|\psi(t_0) - y_0| \leqslant \varepsilon$; we also know to which of the sets E_1 or E_2 the element y_0 belongs.

It is required

a) to construct a model Σ_Δ described by the control system

$$\begin{aligned} &z^{\cdot}(t) = f_\delta(t, \psi_{t_0, t}(\cdot), u(t)), \ t_0 \leqslant t \leqslant \vartheta \\ &z \in H, \ z(t_0) = \psi(t_0), \ f_\delta\colon t \times \psi_{t_0, t}(\cdot) \times u \to H \end{aligned} \tag{2.5}$$

b) to construct the following mapping, dependent on the information error $\varepsilon > 0$, auxiliary parameter ν and partition Δ:

$$U_{\varepsilon, \nu, \Delta}: \{\tau_i, \psi_{t_0, \tau_i}(\cdot), z_{t_0, \tau_i}(\cdot)\} \to L_2([\tau_i, \tau_{i+1}]; U)$$

c) to devise a rule for the selection of ε, ν and Δ such that for any signal ψ the quantity

$$p(u^e(\cdot), U_*) \equiv \inf\left\{\left(\int_{t_0}^{\vartheta} \| u^e(t) - u(t)\|_U^2\, dt\right)^{1/2} \,|\, u(\cdot) \in U_*\right\}$$

will be sufficiently small provided

$$u^e_{\tau_i, \tau_{i+1}}(\cdot) \in U_{\varepsilon, \nu, \Delta}\{\tau_i, \psi_{t_0, \tau_i}(\cdot), z^\delta_{t_0, \tau_i}(\cdot)\},$$

where $U_* \subset L_2([t_0, \vartheta]; U)$ is the set of all controls generating $y_p(\cdot)$, $z^\delta(\cdot) = z(\cdot; \psi(\cdot), u^e(\cdot))$ the trajectory of the model for control $u = u^e(\cdot)$.

A function $z(\cdot) = z(\cdot; \psi(\cdot), u^e(\cdot))$ is a trajectory of Σ_Δ if it is strongly absolutely continuous and satisfies system (2.5) for a.e. $t \in [t_0, \vartheta]$.

3. An algorithm will now be presented for solving the above problem with $U_0 = L_2([t_0, \vartheta]; U)$. As the model Σ_Δ we take system (2.5) with right-hand side

$$f_\delta(t, \psi_{t_0, t}(\cdot), u) = \begin{cases} 0 \text{ for a.e. } t \in [t_0, \tau_j] \\ r^\lambda(t - \delta) + B(t - \delta, \psi(t - \delta))\, u(t) \text{ for a.e. } t > \tau_j \end{cases} \tag{3.1}$$

Here $\lambda > 0, j = 1$ for $y_0 \in E_1$, $j = 2$ for $y_0 \in E_2$, $A_* y = \alpha y$, $\varphi^*(y) = \varphi(y) + {}^1/_2\, |y|^2 + {}^1/_2\, (Ay, y)_{V \times V^*}$ if the signals ψ satisfy inequality (2.3), $A_* y = -Ay$, $\varphi^*(y) = \varphi(y)$ if ψ satisfies inequalities (2.3) and (2.4), $r^\lambda(t) = A_*\psi(t) - \partial\varphi^0(\psi(t)) + f(t)$, if $\varphi^* = \varphi$ and the function $x \to \partial\varphi^0(x)$ satisfies a Lipschitz condition, $r^\lambda(t) = A_*\psi(t) - \partial\varphi_\lambda{}^*(\psi(t)) + f(t)$ otherwise.

We define the mapping $U_{\varepsilon, \nu, \Delta}$ by the rule

$$\begin{aligned} &U_{\varepsilon, \nu, \Delta}\{t_0, \psi(t_0), \psi(t_0)\} = 0 \in L_2([t_0, \tau_j]; U) \\ &U_{\varepsilon, \nu, \Delta}\{\tau_i, \psi_{t_0, \tau_i}(\cdot), z^\delta_{t_0, \tau_i}(\cdot)\} = u^*_{\tau_i, \tau_{i+1}}(\cdot), i \in [j : m - 1] \end{aligned} \tag{3.2}$$

calculating the functions $u^*_{\tau_i, \tau+1}(\cdot)$ at times τ_i by the formulae

$$u^*(t) = \begin{cases} 0, & \text{if } \varkappa_i \leqslant 0 \text{ or } a_i \leqslant \nu\delta^{1/2} \\ \varkappa_i a_i^{-2} h_{i, \psi}(t - \delta) & \text{otherwise} \end{cases}$$

$$\varkappa_i = \left(\pi_i, \int_{\tau_{i-1}}^{\tau_i} r^\lambda(t)\, dt - \chi_i\right), \quad a_i = \left(\int_{\tau_{i-1}}^{\tau_i} \| h_{i, \psi}(\xi)\|_U^2\, d\xi\right)^{1/2}$$

$$\chi_i = \psi(\tau_i) - \psi(\tau_{i-1}), \quad \pi_i = z^\delta(\tau_i) - \psi(\tau_{i-1})$$

$h_{i, \psi}(\cdot)$ is the element of the space $L_2([\tau_{i-1}, \tau_i]; U)$, uniquely defined by the condition: for any $u(\cdot) \in L_2([\tau_{i-1}, \tau_i]; U)$

$$-\int_{\tau_{i-1}}^{\tau_i} (\pi_i, B(\xi, \psi(\xi))\, u(\xi))\, d\xi = \int_{\tau_{i-1}}^{\tau_i} (h_{i, \psi}(\xi), u(\xi))_U\, d\xi$$

$z^\delta(\cdot)$ is the trajectory of the model with initial state $z^\delta(t_0) = \psi(t_0)$ corresponding to the control $u = u^*(t)$.

We shall assume that the following conditions are satisfied.

1°. For any bounded set $K \subset H$ there exists a number $L = L(K)$ such that

$$\| B(t, y) - B(t, z) \|_{L(U,H)} \leqslant L \mid y - z \mid, \ y, z \in K$$

2°. $t \to B(t, y_p(t)) \in L_2([t_0, \vartheta]; L(U, H))$.

3°. If $y_0 \in E_1$, then $\partial\varphi^{*,0}(y_p(t)) \in L_2([t_0, \vartheta]; H)$. If $y_0 \in E_2$, then $\partial\varphi^{*,0}(y_p(t)) \in L_2((t_0, \vartheta]; H)$.

4°. There exists a unique element of minimum norm $u_*(\cdot) \in L_2([t_0, \vartheta]; U)$ with the following property: for a.e. $t \in [t_0, \vartheta]$,

$$y_p^{\cdot}(t) - f(t) - A_* y_p(t) + \partial\varphi^{*,0}(y_p(t)) = \{B_{y_p} u_*(\cdot)\}(t)$$

Consider an arbitrary monotone increasing function $F(\nu)$, $D(F) = [0, +\infty)$, $F(0) = 0$.

Theorem 1. For any $\alpha_0 > 0$ we can find ν_0 and $\delta_0 > 0$ such that, for any number $\nu \in (0, \nu_0)$ and partition Δ of the interval $[t_0, \vartheta]$ of mesh $\delta \leqslant \delta_0$, and for any signal ψ with the above properties,

$$p(u^e(\cdot), U_*) \leqslant \alpha_0 \tag{3.3}$$

provided only that

$$u^e_{\tau_i, \tau_{i+1}}(\cdot) = U_{\varepsilon, \nu, \Delta}\{\tau_i, \psi_{t_0, \tau_i}(\cdot), z^\delta_{t_0, \tau_i}(\cdot)\}, \quad i \in [0 : m - 1] \tag{3.4}$$

$$\max_{i \in [j:m]} \Big(\int\limits_{\tau_{i-1}}^{\tau_i} | \partial\varphi^{*,0}(y_p(\xi)) - \partial\varphi_\lambda^*(y_p(\xi)) |^2 \, d\xi \Big)^{1/2} \leqslant \frac{\nu\sqrt{\delta}}{2} \tag{3.5}$$

$$\varepsilon \leqslant \min\{\nu\delta, {}^1/_2 \nu\lambda\} \tag{3.6}$$

and for $y_0 \in E_2$ the number δ is so small that

$$\delta \int\limits_{t_0+\nu/2}^{\vartheta} | y_p^{\cdot}(\xi) |^2 \, d\xi \leqslant F(\nu) \tag{3.7}$$

Theorem 1 can be proved with the help of the following lemma.

Lemma 1. For any numbers $\nu \in (0, (\vartheta - t_0)/3)$, $\delta \in (0, \nu/4)$ and partition Δ of the interval $[t_0, \vartheta]$ of mesh $\delta(\Delta) \leqslant \delta$, the following estimates hold:

$$\int\limits_{\tau_i}^{\tau_{i+1}} \| u^e(t) \|_U{}^2 \, dt \leqslant \int\limits_{\tau_{i-1}}^{\tau_i} \| u_*(t) \|_U{}^2 \, dt + k\delta^{1/2} | z^\delta(\tau_i) - \psi(\tau_{i-1}) |^2$$

$$| z^\delta(t) - y_p(t) | \leqslant \gamma(\nu, \delta), \ i \in [1 : m - 1], \quad t \in [t_0, \vartheta]$$

if $u^e(\cdot)$ is defined by (3.4), and λ and ε are such that inequalities (3.5) and (3.6) hold.

For $y_0 \in E_1$ we have in Lemma 1 $\gamma = k_1(\nu + \delta + \nu_*(\delta))$, k and k_1 are constants expressible in explicit form and

$$\nu_*(\delta) = \max\Big\{ \int\limits_{\tau_{i-1}}^{\tau_i} \| B(\xi, y_p^{\cdot}(\xi)) \|^2_{L(U,H)} \, d\xi \,|\, i \in [1 : m] \Big\}$$

In the case $y_0 \in E_2$, on the other hand, $\gamma(\nu, \delta)$ is a non-linear function with the property: $\gamma(\nu, \delta) \to 0$ as $\nu \to 0+$, $\delta \to 0+$ and conditions (3.5)-(3.7) are satisfied.

Theorem 2. Let the mapping $x \to \partial\varphi^0(x)$ satisfy a Lipschitz condition. Then the statement of Theorem 1 is true if $u^e(\cdot)$ is defined by (3.4), $\varepsilon \leqslant \delta\nu$, $\varphi^* = \varphi$ and if $y_0 \in E_2$ then condition (3.7) is satisfied.

Under the assumptions of Theorem 2, if $y_0 \in E_1$, the following estimates, which indicate the performance of the algorithm, hold:

$$\max_{t_0 \leqslant t \leqslant \vartheta} \Big| \int\limits_{t_0}^{t} B(\xi, y_p(\xi)) \{u^e(\xi) - u_*(\xi)\} \, d\xi \Big| \leqslant K_1(\delta + \varepsilon + \nu)^{1/2}$$

$$\int\limits_{t_0}^{\vartheta} \| u^e(t) \|_U{}^2 \, dt \leqslant \int\limits_{t_0}^{\vartheta} \| u_*(t) \|_U{}^2 \, dt + K_2(\delta + \varepsilon + \nu)$$

where K_1 and K_2 are constants that can be expressed in explicit form.

Put

$$Q = \{y(\cdot) \in L_2([t_0, \vartheta]; H) \mid y(t) \in \partial\varphi^*(y_p(t)) \text{ for a.e. } t \in [t_0, \vartheta]\}$$

Conditions 3°, 4° are satisfied if $x \to \partial\varphi(x)$ is a single-valued mapping. They are also satisfied if $Q \subset R(B_{y_p})$ and either $A_* y_p(t) \in L_2([t_0, \vartheta]; H)$ or $\varphi(x)$ is the characteristic function of a convex, bounded, closed set.

4. Let $u_p(\cdot)$ be an element of a convex and closed set $P = \{u(\cdot) \in L_2([t_0, \vartheta]; U) \mid u(t) \in P(t) \subset U$ for a.e. $t \in [t_0, \vartheta]\}$, which is unknown at time t_0. At every time $\tau_i \in \Delta$, $i \geqslant 1$, the restriction of the set P to the interval $[\tau_{i-1}, \tau_i]$ becomes known, call it P_i. Put $U_0 = P$, leave the model Σ_Δ as before, and define the strategy $U_{\varepsilon, \nu, \Delta}\{\tau_i, \psi_{t_0, \tau_i}(\cdot), z^{\circ}_{t_0, \tau_i}(\cdot)\} \to \{u_{\tau_i, \tau_{i+1}}(\cdot) \mid u(t) \in P(t-\delta)$ for a.e. $t \in [\tau_i, \tau_{i+1}]\}$ by rule (3.2), calculating $u^*_{\tau_i, \tau_{i+1}}(\cdot)$ as follows:

$$l_i(u^*_{\tau_i, \tau_{i+1}}(\cdot)) = \min\{l_i(u_{\tau_i, \tau_{i+1}}(\cdot)) \mid u_{\tau_i, \tau_{i+1}}(\cdot) \in P_i\} \tag{4.1}$$

$$l_i(u_{\tau_i, \tau_{i+1}}(\cdot)) = \Big(\pi_i, \int_{\tau_{i-1}}^{\tau_i} B(\xi, \psi(\xi)) u(\xi+\delta)\, d\xi\Big) + \nu \int_{\tau_i}^{\tau_{i+1}} \|u(\xi)\|_U{}^2\, d\xi$$

Theorem 3. Under the assumptions of Theorem 2, assume that conditions 1°–4° are also satisfied, with the proviso that $u_*(\cdot)$ in condition 4° is the unique element of minimum norm in P. Then there exist monotone increasing functions $\delta(\nu)$ and $\varepsilon(\nu, \delta)$, $\delta(0) = \varepsilon(0, 0) = 0$, $D(\delta(\cdot)) = [0, +\infty)$, $D(\varepsilon(\cdot, \cdot)) = [0, +\infty) \times [0, +\infty)$, such that for any $\alpha_0 > 0$, $\nu \in (0, \nu_0)$, $\delta = \delta(\Delta) \in (0, \delta(\nu))$, $\varepsilon \in (0, \varepsilon(\nu, \delta))$ and any signal ψ with the properties described in Sect.2, inequality (3.3) is true if $\nu_0 = \nu_0(\alpha_0)$ is sufficiently small, and the strategy $U_{\varepsilon, \nu, \Delta}$ is defined in accordance with (3.2), (4.1).

A similar theorem can be proved for the case in which $U_0 = P$ and the assumptions of Theorem 1 are satisfied.

By known results of the theory of accretive functions /7/, for any $\nu > 0$ and $\delta > 0$ there exists $\lambda_0 > 0$ such that inequality (3.5) is true for all $\lambda \in (0, \lambda_0)$.

Let us assume that $\varphi(y) = I_K(y)$ is the characteristic function of a convex closed set $K \subset H$. Inequality (2.1) is rewritten

$$(y^{\cdot}(t) - B(t, y(t)) u(t) - f(t), y(t) - z) + (Ay(t), y(t) - z)_{V \times V^*} \leqslant 0 \text{ for a.e. } t \in (t_0, \vartheta) \text{ and all } z \in K$$

If the signals ψ possess properties (2.3), (2.4), we can put $r^\lambda(t) = -A\psi(t) + f(t)$ in (3.1). Otherwise,

$$r^\lambda(t) = \alpha y - \partial\Phi_\lambda(\psi(t)) + f(t), \quad \Phi(y): H \to \bar{R}$$

$$\Phi(y) = \begin{cases} {}^1/_2 (Ay, y)_{V \times V^*} + {}^1/_2 \alpha |y|^2, & y \in V \\ +\infty, & y \in H \setminus V \end{cases}$$

5. The control process occurring in a thermostat regulated by the temperature in a region $\Omega \subset R^3$ /8/, is formalized as a variational inequality (2.1), in which Ω is a bounded domain with fairly smooth boundary (e.g., of class C_2), $y(x, t)$ is the temperature of the body occupying the region Ω, $u(x, t)$ is the controllable thermal flux and $H = L_2(\Omega)$, $V = H_0{}^1(\Omega)$, $B(t, y)u = u$,

$$A: H_0{}^1(\Omega) \to H^{-1}(\Omega): (Ay, z)_{V \times V^*} = \sum_{i,j=1}^{3} \int_\Omega a_{ij}(x) y_{x_i} z_{x_j}\, dx + \int_\Omega a_0(x) y(x) z(x)\, dx$$

$$a_0(\cdot) \in L_\infty(\Omega), \quad a_{ij}(\cdot) \in C^1(\bar{\Omega}), \quad a_{ij} = a_{ij}, \quad i \in [1:3]$$

$a_0(x) \geqslant \mu > 0$ for a.e. $x \in \Omega$, for some $\omega > 0$

$$\sum_{i,j=1}^{3} a_{ij}(x) \xi_i \xi_j \geqslant \omega \|\xi\|^2_{R^3} \text{ for all } \xi \in R^3 \text{ and a.e. } x \in \Omega$$

$$\varphi(y) = \int_\Omega g(y(x))\, dx, \quad \partial g(r) = \beta(r) = \begin{cases} a_1(r - \vartheta_1), & -\infty < r < \vartheta_1 \\ 0, & r \in [\vartheta_1, \vartheta_2] \\ a_2(r - \vartheta_2), & \vartheta_2 < r < +\infty \end{cases}$$

$a_1, a_2, \vartheta_1, \vartheta_2$ are constants.

The problem of dynamically determining the thermal flux $u(x, t)$ given the temperature of the body measured at times τ_i is solved using the algorithms proposed in Sects.3 and 4 above. If the approximate value of the temperature $y(x, t)$, say $\psi(x, t)$, satisfies (2.3), (2.4), then the model Σ_Δ is described by one of the systems

$$y_t^{\cdot}(x,t) = -A_0\psi(x,t-\delta) - \beta(\psi(x,t-\delta)) + u(x,t)$$

$$y_t^{\cdot}(x,t) = -A_0\psi(x,t-\delta) - \lambda^{-1}\{\psi(x,t-\delta) - J_\lambda\psi(x,t-\delta)\} + u(x,t)$$

Here $J_\lambda\psi(x) = w(x) \in L_2(\Omega)$ is such that

$$w(x) = \psi(x) - \lambda\beta(w(x))$$

$$A_0 y(x) = -\sum_{i,j=1}^{3}(a_{ij}(x)\,y_{x_i}(x))_j + a_0(x)\,y(x)$$

If $\psi(x,t)$ satisfies only inequalities (2.3), then Σ_Δ has the form

$$y_t^{\cdot}(x,t) = -\lambda^{-1}\{\psi(x,t-\delta) - J_\lambda(\psi(x,t-\delta))\} + u(x,t)$$

$J_\lambda\psi(x) = w(x) \in H_0^1(\Omega) \cap H^2(\Omega)$ is a solution of the equation

$$w(x) = \psi(x) - \lambda\{A_0 w(x) + \beta(w(x))\} \text{ a.e. on } \Omega$$

6. Let V, U and H be Sobolev spaces on Ω, $B(t,y)u = u$, and A the Laplace operator. In a computer solution of the control-modelling problem, one naturally replaces Ω by a suitable grid $\bar{\omega}_h = \{x_j \mid j = 1,\ldots,N\}$ (/9/, p.69) with step size h and assumes that the values of $\psi(x,\tau_i)$ are measured at the grid-points x_j. Eq.(2.5) must then be replaced by a difference equation

$$z^\delta(x_j,\tau_{i+1}) = z^\delta(x_j,\tau_i) + \delta\{r_1^\lambda(x_j,\tau_{i-1}) + u(x_j,\tau_i)\}, \quad i \geqslant 1$$

and the function $u(\cdot) = u^*_{\tau_i,\tau_{i+1}}(\cdot)$ by a difference function $u = u^*(x_j,\tau_i) = 0$ if $\varkappa_{1i} \leqslant 0$ or $a_{1i} \leqslant \nu\delta^{1/4}$, $u = u^*(x_j,\tau_i) = \varkappa_{1i}\,a_{1i}^{-2}h_{i,\psi}(x_j)$ otherwise. Here $a_{1i}, r_1^\lambda(x_j,\tau_{i-1}), \varkappa_{1i}$ are difference analogues of a_i, $r^\lambda(\tau_{i-1}), \varkappa_i, h_{i,\psi}(x_j) = z^\delta(x_j,\tau_i) - \psi(x_j,\tau_{i-1})$. The values of $A\psi(\tau_i)$ must then also be measured, whenever possible, at the grid-points x_j. If $A\psi(\tau_i)$ cannot be measured at x_j, they may be approximated by suitable difference relations, e.g., by the expression $L_h\psi$ /9/ when $\Omega = (a,b) \subset R^1$.

The process of calculating the load $g(x,t)$ applied to a membrane was modelled on the computer, with

$$F = 1,\ x = (x_1,x_2) \in R^2,\quad \Omega = (0,1)\times(0,1)$$

$$t_0 = 0,\quad \vartheta = 1,\quad \nu = 10^{-4}$$

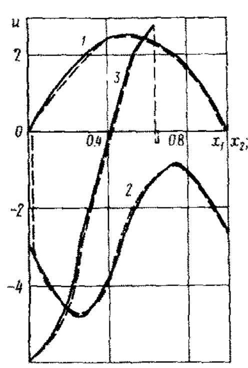
$$y_p(x,t) = \begin{cases} -10x_1(x_1-\alpha(t))^2 x_2(1-x_2) & x_1 \leqslant \alpha \\ 0, & x_1 > \alpha \end{cases}$$

$$u_*(x,t) = \begin{cases} \partial y_p(x,t)/\partial t - \Delta y_p(x,t), & x_1 \leqslant \alpha \\ 0, & x_1 > \alpha \end{cases}$$

$$\alpha = {}^1/_2 + {}^1/_4 \sin 2\pi t,\quad y_0(x) = y_p(x,0)$$

The phase trajectory of the model $z^\delta(\cdot)$ was calculated by Euler's method with step size $\delta = 0.002$. The region Ω was divided into squares with side 0.02 and replaced by a uniform grid $\bar{\omega}_h$ of step size $h = 0.02$ /9/. Formation of the control $u^e(t,x_j)$ used the values of $\psi(x_j,\tau_i)$ and $\Delta\psi(x_j,\tau_i)$ at the grid-points of $\bar{\omega}_h$ only.

The figure shows sections by hyperplanes $\Gamma_1,\Gamma_2,\Gamma_3$ of the load $g(x,t) = u_*(x,t)$ (the solid curves 1, 2, 3) and sections found at $\psi(x,\tau_i) = y_p(x,\tau_i) + 10^{-5}(x_1^2 + x_2^2)$, using the algorithm (3.1), (3.2), (3.4), of the load $u^e(x,t)$ (the dashed curves 1, 2, 3). We had $p(u^e(\cdot),U_*) = 0.221$ and the hyperplane were defined as follows:



$$\Gamma_1 = \{(x,t) \mid x_1 = 0.6,\ \ t = 0.4\}$$

$$\Gamma_2 = \{(x,t) \mid x_1 = 0.08,\ \ x_2 = 0.8\}$$

$$\Gamma_3 = \{(x,t) \mid x_2 = 0.6,\ \ t = 0.4\}$$

Remark. Referring to the problem of the deflection of a membrane, let us explain the mechanical meaning of inequalities (2.4) and the conditions $y_0 \in E_1, y_0 \in E_2$. Since every difference function $\psi(x_j,\tau_{i-1})$ may be defined at the other points of Ω so as to obtain a function $\psi = \psi_h(x,\tau_{i-1})$, inequalities (2.4) imply restrictions on the smoothness of ψ_h and the quality of the approximation of y_p by this function. For example, the function $\psi_h(x,\tau_{i-1})$, with $\psi(x,t) = \psi_h(x,\tau_{i-1})$, $t \in [\tau_{i-1},\tau_i]$, must have a second derivative with respect to the space variable, which is a good approximation to $\Delta y_p(x,t)$ on the average in the interval $[\tau_{i-1},\tau_i]$. Inequalities (2.4) may be treated as conditions on the frequency at which the deflection is measured. This is particularly evident if the Laplace operator is approximated by a difference operator, such as $L_h\psi$, and the left-hand side of (2.4) is replaced by a suitable difference expression.

The condition $y_0 \in E_1$ means that the role of the function $y_0(x)$ describing the initial deflection is played by a function with certain "smoothness" properties ($y_0(x) \in H_0^1(\Omega)$). The condition $y_0 \in E_2$ corresponds to the case in which the deflection at time t_0 is described by a function in $L_2(\Omega)$.

REFERENCES

1. KRASOVSKII N.N. and SUBBOTIN A.I., Positional Differential Games, Nauka, Moscow, 1974.
2. OSIPOV YU.S., On the theory of differential games in systems with distributed parameters. Dokl. Akad. Nauk SSSR, 223, 6, 1975.
3. KRYAZHIMSKII A.V. and OSIPOV YU.S., On modelling of the control in a dynamic system. Izv. Akad. Nauk SSSR, Tekh. Kibernetika, 2, 1983.
4. OSIPOV YU.S. and KRYAZHIMSKII A.V., On the dynamic solution of operator equations. Dokl. Akad. Nauk SSSR, 269, 3, 1983.
5. MAKSIMOV V.I., Positional modelling of controls and initial functions for Volterra systems. Differents. Urav., 23, 4, 1987.
6. GLOWINSKI R., LIONS J.-L. and TRÉMOLIÈRES R., Analyse numérique des inéquations variationelles, Vols.1-2, Paris, Dunod, 1976.
7. EVANS L.S., Non-linear evolution equations in an arbitrary Banach space. Israel J. Math., 26, 1, 1977.
8. DUVAUT G. and LIONS J.L., Les inéquations en mécanique et en physique. Paris, Dunod, 1962.
9. SAMARSKII A.A., Theory of difference schemes, Nauka, Moscow, 1977.

Translated by D.L.

PMM U.S.S.R.,Vol.52,No.5,pp.585-592,1988
Printed in Great Britain

0021-8928/88 $10.00+0.00

MULTILAYER FLOWS OF AN INCOMPRESSIBLE LIQUID OVER AN UNEVEN BOTTOM UNDER THE ACTION OF SURFACE PRESSURE*

K.A. BEZHANOV

The plane problem of the shear flow of an ideal heavy incompressible stratified liquid of finite depth over an uneven bottom is studied. The liquid has a finite number of layers and the stratification at their boundaries is discontinuous. An exact non-linear integrodifferential equation is obtained describing the internal and surface waves generated by the irregularities of the bottom, and by surface pressure. The basic properties of the spectrum of the linear problem proved in /1/, which generalize the results of /2, 3/, are formulated. A solution of the linear problem is obtained in the form of a Fourier series in terms of the eigenfunctions corresponding to the integral Fredholm equation or of the equivalent boundary value problem. The case of resonant reinforcement of the corresponding mode is discussed for the mean stream velocities close to, but smaller than the critical velocity. A non-linear problem of a streamlined flow with the formation of an internal two-soliton wave is considered for the case in which the mean stream velocities are close to and larger than the critical velocity.

1. Derivation of the basic equations. We consider the plane, steady-state flow of an ideal heavy incompressible stratified liquid above an uneven bottom, in the case when a known pressure is applied to the free surface of the liquid. The x axis is directed along the horizontal level of the bottom, and the y axis is directed vertically upwards. A one-dimensional shear flow is specified as $x \to -\infty$, with stable discontinuous stratification. When a one-dimensional stratified flow is acted upon by a known surface pressure $p_0(x)$ and by the irregularities of the bottom $y_0(x)$, it generates a two-dimensional stratified flow, the functions $p_0(x)$ and $y_0(x)$ are assumed to be continuous and finite, and the segment $[-x_0, x_0]$ is their common carrier. The liquid consists of n layers, the density and tangential

**Prikl.Matem.Mekhan.*,52,5,751-759,1988